Majority Judgment Theory and Paradoxical Results

Manzoor Ahmad Zahid

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Abstract

The majority judgment theory is developed by Michel Balinski and Rida Laraki for social choice [2, 3]. In this paper we will show that majority judgment theory does not always yield a majority choice, is winner and rank inconsistent and the median based majority judgment may yield paradoxical results.

1 Introduction

In this paper we give an analysis of majority judgment theory. In the first section we explain how the majority judgment theory works. In the second section we present interesting tie braking rules. In the third section we derive some paradoxical results with examples.

2 The basic model of majority judgment

The traditional framework of social choice theory, based on rankings of the alternatives by voters, is riddled with impossibility theorems, saying roughly that a social ranking function or choice function with only nice properties cannot exist [1]. A new theory of voting is proposed by Balinski and Laraki [2], which is quite similar to range voting but using the median instead of the average and with an interesting tie braking rule. It allows the voters to convey more information in the input of the choice and ranking functions. A common language for grading is introduced and much more information of the jury members is asked. As in the old framework, there is a finite set C of m competitors, $a_1, a_2, \ldots, a_m$, and a finite set J of n judges, $1, 2, \ldots, n$. Furthermore, a common language L is a set of strictly ordered grades r_n, which may be finite or an interval of the real numbers. We take $r_i \geq r_j$ to mean that r_i is a higher grade than r_j or $r_i = r_j$. The input for a social grading function SGF is then an m by n matrix, called a profile, filled with grades (i.e., elements of L), meaning that every competitor (m rows) is given a grade by every judge (n columns). A method of grading is a function F that assigns to every profile one output, called the final grade, in the same language, for every competitor.

$F : L^{m \times n} \to L^m$, is defined by

$$\begin{array}{cccc} 1 & 2 & \ldots & \text{n} \end{array}$$
$$\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix} \to (f(r_{11}, r_{12}, \ldots, r_{1n}), \ldots, f(r_{m1}, r_{m2}, \ldots r_{mn}))$$

where $f : L^n \to L$ and $f(r_{i1}, r_{i2}, \ldots, r_{in})$ is the final grade of competitor a_i.

Balinski and Laraki [2] consider order functions; the kth order function f^k takes as input an n-tuple of grades and gives as output the kth highest grade. They show that these order functions are demonstrably best for aggregating. They have used middlemost aggregation functions. Suppose $r_1 \geq r_2 \geq \ldots \geq r_n$. A middlemost aggregation function f is defined by $f(r_1, r_2, \ldots, r_n) = r_{(n+1)/2}$ when n is odd, and $r_{n/2} \geq f(r_1, r_2, \ldots, r_n) \geq r_{(n+2)/2}$ when n is even. So, when n is odd, the order function $f^{n/2}$ is the middlemost aggregation function. When n is even, there are infinitely many. In particular, the upper-middlemost, defined by $f^{n/2}(r_1, r_2, \ldots, r_n) = r_{n/2}$, and $f^{(n+2)/2}$ the lower-middlemost aggregation function, defined by $f^{(n+2)/2}(r_1, r_2, \ldots, r_n) = r_{(n+2)/2}$. Any grade not bounded by the middlemost aggregation functions is condemned by an absolute majority of judges as either too high or too low. $f(r_{i1}, r_{i2}, \ldots, r_{in})$ is called the *majority grade* of candidate a_i.
The majority grade of a candidate is his or her median grade. It is simultaneously the highest grade approved by a majority and the lowest grade approved by a majority. For instance, if a candidate has got the grades, 10, 8, 7, 5, 2 on a scale of 1 till 10, his majority grade will be 7, since there is a majority of judges who think the candidate should have at least grade 7, and an other majority of judges who think the candidate should have at most grade 7. Let p be the number of grades above the majority grade, and q be the number of grades below the majority grade. Then

$$\alpha^* = \begin{cases} \alpha^+ & \text{if} \quad p > q; \\ \alpha^\circ & \text{if} \quad p = q; \\ \alpha^- & \text{if} \quad p < q \end{cases}$$

When the number of voters is odd, the majority grade is the median, or the one middle grade. In the case of an even number of voters and a candidate's two middle grades are different, Balinski and Laraki argue that the lower of the two middle grades must be the majority-grade.

3 Majority Judgment tie-breaking rules

In [2] the majority grades of the candidates are used to calculate the majority ranking. The general majority ranking $>_{maj}$ between two competitors is determined as follows:

- if $f^{maj}(A) > f^{maj}(B)$, then $A >_{maj} B$.

- If $f^{maj}(A) = f^{maj}(B)$, the majority grade is dropped from the grade of each competitor and the procedure is repeated.

In [3] Balinski and Laraki present the following tie braking rule for the case of large elections. Three values attached to a candidate, called the candidate's *majority value*, are sufficient to determine the candidate's place in the majority ranking:

$$(p, \alpha, q) \text{ where } \begin{cases} p = \text{percentage of grades above the majority grade} \\ \alpha = \text{majority grade, and} \\ q = \text{percentage of grades below the majority grade} \end{cases}$$

In [3] the order between two majority values is now defined as follows:

$$(p, \alpha^*, q) > (s, \beta^*, t) \text{ if } \alpha^* > \beta^*$$

where $\alpha^* > \beta^*$ if $\alpha > \beta$ and $\alpha^+ > \alpha^\circ > \alpha^-$.

Now suppose α^* and β^* are the same. Then

$$(p, \alpha^+, q) > (s, \alpha^+, t) \text{ if } \begin{cases} p > s, \text{or} \\ p = s \text{ and } q < t \end{cases}$$

$$(p, \alpha^-, q) > (s, \alpha^-, t) \text{ if } \begin{cases} q < t, \text{or} \\ q = t \text{ and } p > s \end{cases}$$

$$(p, \alpha^\circ, q) > (s, \alpha^\circ, t) \text{ if } p < s, \text{ where } p = q \text{ and } s = t.$$

The tie braking rules in [2] and [3] are different. Consider a simple example:

	Excellent	V.Good	Good	Accepted	Poor	Rejected	Total
A	17	20	**24**	0	16	23	100
B	16	20	**25**	21	17	1	100

A and B have both majority grade *good*$^-$. According to the simplified tie braking rule in [3], A is the winner. If we apply the general tie breaking rule in [2], then B turns out to be the winner.

Consider, for instance, a similar example for a large election:

	Excellent	V.Good	Good	Accepted	Poor	Rejected	Total
A	18758	25242	**35984**	10016	9126	874	100000
B	29818	14182	**34952**	11048	9478	522	100000

A's majority value is $(44000, good, 20016)$ and his majority $grade^*$ is $good^+$. B's majority value is $(44000, good, 21048)$ and his majority $grade^*$ is also $good^+$. A and B have the same grades, so, by the tie breaking rule in [3], the winner is A. However, when we apply the general tie breaking rule in [2], we have B as winner. These examples clearly show that the tie breaking rules in [2] and [3] may give different results.

4 Paradoxes in majority judgment theory

The voting paradoxes have an important role in social choice theory. The founding fathers of social choice theory, Marquis de Condorcet (a French mathematician, philosopher, economist, and social scientist), and Jean-Charles de Borda (a French mathematician, physicist, political scientist, and sailor), were aware of some paradoxes. The importance of the voting paradoxes was realized after Kenneth Arrow's impossibility theorem. The essence of this theorem is that there is no method of aggregating individual preferences over three or more alternatives that satisfies several conditions of fairness and always produces a logical result. Michel Balinski and Rida Laraki gave a new voting system, assuming a common language, and using two different tie breaking rules, which unfortunately, also seem to produce paradoxes. Suppose there are two alternatives and one hundred judges, whose evaluations are summarized in the following table.

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	50	9	41	**50**	0	0	0	0
B	4	4	**47**	3	5	12	29	49

According to Balinski's majority judgment theory [2], A's majority value is good and B's majority value is very good. B wins, although twenty nine judges rejected him. On the other hand all the judges gave to A the judgment good to excellent. Now consider the cumulative majority judgment grades shown in the table below, derived from the table above.

At Least	Excellent	V.Good	Good	Accepted	Poor	Rejected
A	9	50	100	100	100	100
B	4	51	54	59	71	100

In the light of the cumulative majority judgment grades, it is crystal clear that A has a majority, but B wins. Is this rational? According to the Borda count [4], the Borda score of A is 359, and of B only 239.

Theorem 1: There exists a middlemost aggregation choice which is not a majority choice.

Consider the following example, in which there are two alternatives and one hundred judges, whose judgments are summarized in the following table.

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	1	1	**98**	1	0	0	0	1
B	50	50	**1**	0	0	0	49	49

According to both tie breaking rules in [2] and [3], B wins. We can see, 99 percent of the judges give to A the judgment very good to excellent and only 51 percent of the judges give to B the judgment very good to excellent. We can see that the Borda count [4] of A is 400, and of B only 254. According to the Borda score, A will win. Does B have a majority? Now consider the cumulative majority judgment grades shown in the table below, derived from the table above.

at least	Excellent	V.Good	Good	Accepted	Poor	Rejected
A	1	99	100	100	100	100
B	50	51	51	51	51	100

The cumulative majority judgment grades show that A should win, not B. In the light of these examples we can not claim that Balinski and Laraki's majority judgment theory always selects a majority choice.

Majority judgment theory can produce unacceptable results for small choices, like figure skaters, gymnastic competition, pianists competition, wine competition etc. Consider a shooting contest with five players and one judge; they played ten rounds and the following grades are obtained.

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	5	5	0	0	0	0	**5**	0
B	5	0	5	0	0	0	**5**	0
C	5	0	0	5	0	0	**5**	0
D	5	0	0	0	5	0	**5**	0
E	0	0	0	0	0	**6**	4	4

E is the winner in this contest instead of A who has five times grade excellent. A's median grade should be good, but Balinski and Laraki take the lowest median grade, which causes the loser-is-winner paradox [4, 5].

Dan Bishop [7] and Rob Lanphier[8] have also pointed out the following problem: in simple median based range voting, election results can be changed if zero ballots are added. As an example consider the following election with range $0 - 6$.

A's scores are $1, 2, 4, 4, 6$ $Median = 4$

B's scores are $2, 3, 3, 6, 6$ $Median = 3$

If a zero ballot "$A = 0, B = 0$" is added then

A's scores are $0, 1, 2, 4, 4, 6$ $Median = 2$

B's scores are $0, 2, 3, 3, 6, 6$ $Median = 3$

According to the theory, when the number of voters is even, then the majority grade is the lower middle grade. After adding a zero ballot the winner has changed. Also suppose that two friends Romeo and Julia decided not to cast their votes because Romeo is in favor of A and wants to give A one grade higher than B i.e., 6 and 5 respectively and Julia is in favor of B and wants to give B one grade higher than A , i.e., 6 and 5 respectively. Are their votes tied ?

Let us see what happens if we add their votes to our example above. Then B becomes the winner instead of A.

A's scores are $1, 2, 4, 4, 5, 6, 6$ $Median = 4$

B's scores are $2, 3, 3, 5, 6, 6, 6$ $Median = 5$

What happens if Romeo and Julia give the highest same grade 6 to both ? Then

A's scores are $1, 2, 4, 4, 6, 6, 6$ $Median = 4$

B's scores are $2, 3, 3, 6, 6, 6, 6$ $Median = 6$

Now again B is the winner instead of A.

Grade consistent: If there are two separate parts of an electorate and the majority grade of a candidate in each is α, then the majority grade of the candidate is α in the whole electorate as well.

Theorem 2: The majority judgment theory is grade consistent but neither rank nor winner consistent.

Proof: Suppose the majority grade of a candidate A is α in both districts I and II, that is, in district-I there is a majority M_1 that judges A deserves at least grade α and another majority M_2 that judges A deserves at most grade α. Similarly, in district-II there are majorities N_1 and N_2. Then in the union of I and II, $M_1 \cup N_1$ is a majority that judges that A deserves at least grade α, and $M_2 \cup N_2$ is a majority that judges that A deserves at most grade α. Thus α is the majority grade of A in the whole electorate.

The majority judgment theory shows the two districts paradox [6]. If there are two separate parts of an electorate and one candidate is winner in both electorates, then any other may be the winner in the whole electorate. For example, four candidates are contesting in the two districts, in district-I, fifty judges gave their judgment as follows:

District-I

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	25	10	15	**13**	9	2	1	12
B	20	2	5	13	**15**	8	7	15
C	14	1	6	7	**22**	7	7	14
D	11	11	**15**	1	2	5	16	24

Candidate D has a majority grade above all others, so he is the winner in district-I and the ranking order is $D > A > B > C$. Now consider the other part of the electorate in which eighty judges gave their judgments as follows:

District-II

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	35	10	25	**10**	18	10	7	35
B	39	6	8	12	13	**24**	17	17
C	39	5	10	18	6	**23**	18	18
D	33	11	22	**15**	14	12	6	32

Both A and D have the same grade *Good*. By the tie braking rule in [2], D is the winner in this district-II and the ranking order is $D > A > B > C$. Next, let us combine the votes from district-I and district-II.

Combined District-I and District-II results:

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	60	20	40	**23**	27	12	8	47
B	46	8	13	25	**28**	32	24	56
C	47	6	16	25	**28**	30	25	55
D	59	22	37	**16**	16	17	22	55

When we combine district-I and district-II, we see that the majority judgment is rank and winner inconsistent. A's majority value is $(60, good, 47)$ and his majority *grade** is *Good*$^+$. D's majority value is $(59, good, 55)$ and his majority *grade** is also *Good*$^+$. Both A and D have the same majority grades. By the tie braking rules of both [2] and [3], A is the winner in the whole electorate instead of D who is the winner in the two separate districts; this is the two districts paradox . The ranking order is $A > D > C > B$, which is not the same as in district-I and district-II, thus the majority judgment theory is neither rank nor winner consistent.

Consider another simple example for two candidates A and B with range voting 1-6, using the median, in two districts, which is also inconsistent.

In District-I:

A's scores are $6, 4, 3, 3, 1$ Majority grade =3

B's scores are $6, 6, 2, 2, 1$ Majority grade =2

In District-II:

A's scores are $6, 6, 5, 2, 2$ Majority grade =5

B's scores are $6, 6, 4, 4, 1$ Majority grade =4

So A wins in both districts.

In the combined Districts I and II:

A's scores are $6, 6, 6, 5, 4, 3, 3, 2, 2, 1$ Majority grade = 3

B's scores are $6, 6, 6, 6, 4, 4, 2, 2, 1, 1$ Majority grade= 4

A wins in each district I and II, while B wins when both districts are joined together. By the way, Balinski and Laraki claim in [2] that one can not expect that the majority judgment is winner consistent.

Majority judgment is not monotonic, more precisely, increased support for a winner may turn him into a loser. Suppose that a society has ten members, they all are invited for dinner at a restaurant. They are offered two different menus and there is a big discount to take the same menu. They decided to choose the best menu by majority judgment. At the time of voting, one member was late and nine members were present. They gave their judgments as follows:

Menu	Excellent	V.Good	Good	Accepted	Poor	Rejected
A	1	2	3	3	0	0
B	0	4	1	1	1	2

According to the tie braking rule in [2], menu A is selected by majority judgment. They wanted to place the order for menu A, when the last member arrived. He was also asked which menu he would like. The $10th$ member, gave as his judgment that menu A is excellent and menu B is very good. He is also in favor of menu A. What happens, if we include his judgment. Will the menu change? Yes, menu B will be selected instead of menu A. May be, this is due to the fact that Balinski and Laraki take the lower median, in case there is more than one median.

5 Conclusion

Balinski and Laraki use median based range voting. The common language of the ballot paper seems to be an improvement with respect to the old framework of Arrow, but the tie braking rules which are different in [2] and [3], produce different results and may yield paradoxical results.

References

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